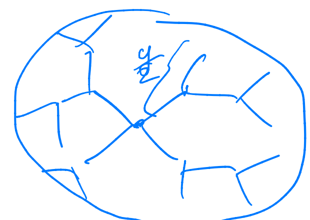
Lecture 9: Probabilistic Method

Theorem 1 (Markov inequality). *For a nonnegative random variable and $a > 0$, it holds:*

$$\Pr(X \geq a) \leq \frac{\mathbf{E}(X)}{a}.$$

1: Prove Markov's inequality (fine if X has finitely many values)

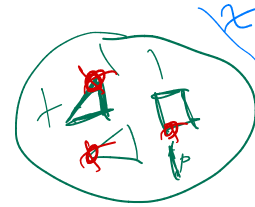
$$E(X) = \sum_i P(X=i)\, i \geq \sum_{i \geq a} P(X=i)\, a = a \cdot P(X \geq a)$$



Theorem 2 (Erdős, 1959). *For every pair of integers g and k, there exists a graph G with girth $g(G) > g$ and chromatic number $\chi(G) > k$.*

$\chi(G(n,p))$ BIG

$\alpha(G(n,p))$ SMALL



Let us use the following notation for the falling factorial

$$(n)_i := \binom{n}{i} i! = n(n-1)(n-2)\cdots(n-i+1).$$

Proof. Proof idea: Take a random graph $\mathcal{G}(n,p)$. Show it has few short cycles and small independent set. Small independent set means high chromatic number. Remove a vertex for each short cycle such that at least 1/2 of the vertices remain. Now no short cycles and still small independent set.

Let $0 < \beta < 1/g$ and let G be from $\mathcal{G}(n,p)$, where $p = n^{\beta-1}$. $= \frac{1}{n^{1-\beta}}$
Let X be the random variable that counts the number of cycles of length $\leq g$.

IF α SMALL χ BIG

$\chi \geq \frac{n}{\alpha}$

2: Show that $\mathbf{E}(X) < n/4$ for sufficiently large n.

$$E(X) \leq \sum_{i=3}^{g} n^i p^i = \sum_{i=3}^{g} n^i \cdot n^{i(\beta-1)} = \sum_{i=3}^{g} n^{i\beta} \leq g\, n^{g\beta} \in o(n)$$

$(\frac{1}{2})^i = p^i$

$X = \sum P(C \text{ APPEARS})$

CYCLE C OF LENGTH AT MOST g

$\beta < \frac{1}{g}$ $\qquad < \frac{n}{4}$

$\boxed{\text{FOR LARGE } n}$

3: Use Markov inequality to find an upper bound probability that the number of cycles of length $\leq g$ is at least $n/2$.

$$P\left(X \geq \frac{n}{2}\right) < \frac{n/4}{n/2} = \frac{1}{2}$$

Now, let $\gamma = \lceil \frac{3}{p} \ln n \rceil$.

4: Show that $\Pr(\alpha(G) \geq \gamma) < \frac{1}{2}$ for sufficiently large n. Actually, it goes to zero as $n \to \infty$.
Hint: $(1-p)^x \leq e^{-px}$ for any $x > 0$.

$$\Pr(\alpha(G) \geq \gamma) \leq \binom{n}{\gamma} (1-p)^{\binom{\gamma}{2}} < n^{\gamma} e^{-p\gamma(\gamma-1)/2} =$$

$$= n^{\gamma} e^{-\cancel{p}\cdot(\frac{3}{\cancel{p}}\ln n)(\gamma-1)/2} = n^{\gamma} n^{-3(\gamma-1)/2} = n^{-\frac{\gamma}{2}+3/2} \to 0$$

As $n \to \infty$

$< 1/2$ FOR LARGE n

Now, let n be large enough such that above two bounds are $< 1/2$. Then, there exists a graph G with less than $n/2$ cycles of length $\leq g$ and with independent set smaller than $3n^{1-\beta} \ln n$. From each cycle of length $\leq g$ of G remove a vertex, this way we obtain a graph G' on at least $n/2$ vertices, whose girth is $> g$ and with independent number $\alpha(G') \leq \alpha(G)$.

5: Calculate the chromatic number of G' and finish the proof.

$$\chi(G') \geq \frac{|V(G')|}{\alpha(G')} \geq \frac{n/2}{3n^{1-3}\ln n} = \frac{n^3}{6\ln n} \nearrow \text{TO } \infty$$

As $n \to \infty$

TAKE n LARGE ENOUGH TO SATISFY $\frac{n^3}{6\ln n} > K$

$\square$

1 Lovász Local Lemma

Let A_i be bad events and we want to avoid all of them.

In practice the basic method is often not useful as the sum $\sum \Pr(A_i)$ is very often bigger than 1.

In general

$$\Pr(A_1 \cup A_2 \cup \cdots \cup A_n) \le \sum_i \Pr(A_i),$$

which can easily be more than 1.

6: Assume that all $A_1, \ldots, A_n$ are independent and nontrivial, i.e. $0 < \Pr(A_i) < 1$. Improve the upper bound in the union of A_i by considering $\bar{A}_i$.

$$P(A_1 \cup A_2 \cup \ldots \cup A_n) = 1 - P(\bar{A}_1 \cap \bar{A}_2 \cap \ldots \bar{A}_n)$$
$$= 1 - P(\bar{P}_1) . P(\bar{A}_2) \ldots P(\bar{A}_n) < 1$$

$P(A \cap B) = P(A) . P(B)$

Lovász Local Lemma extends this when events are mostly independent.

An event A is **mutually independent** from the events $B_1, \ldots, B_k$, if for each subset $J \subseteq \{1, 2, \ldots, k\}$, it holds:

$$\Pr(A \cap \bigcap_{j \in J} B_j) = \Pr(A) \Pr(\bigcap_{j \in J} B_j).$$

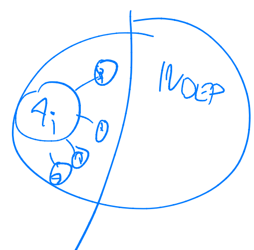
Note: It is stronger than A being independent with each B_i individually!

Theorem 3 (Symmetric Lovász Local Lemma)**.** *Let $A_1, A_2, \ldots, A_n$ be events for which $\Pr(A_i) \le p$. Suppose that each A_i mutually independent of a set of all A_j's but except of at most other d of them. If*

$$ep(d+1) \le 1$$

then

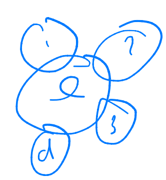
$$\Pr\Big(\bigcap_{i=1}^{n} \bar{A}_i\Big) > 0.$$



A variation of this result replaces the assumption $ep(d+1) \le 1$ by $4pd \le 1$. Note that n does not matter in the lemma.

A coloring of a hypergraph is proper if no edge is monochromatic.

Theorem 4. *Let $\mathcal{H}$ be a hypergraph such that every edge contains at least k vertices and it is incident with at most d other edges. If $e(d+1) \le 2^{k-1}$, then $\mathcal{H}$ 2-colorable.*



Proof. Color the vertices of $\mathcal{H}$ uniformly and independently by red and blue.

7: Finish the proof

RANDOMLY COLOR EACH VERTEX

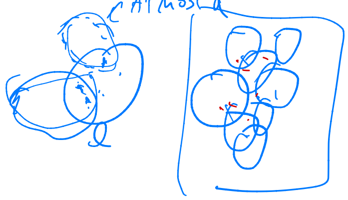
A_e .. EDGE e IS MONOCHROMATIC

$P(A_e) \le 2 \cdot \left(\frac{1}{2}\right)^k = 2^{1-k}$

$e \cdot p(d+1) \le 1$ (2)

A_e IS MUTT. INDEPENDENT OF ALL BUT .. (AT MOST) d OTHER EVENTS

$e \cdot p(d+1) \le 2^{k-1} \cdot 2^{1-k} = 1$



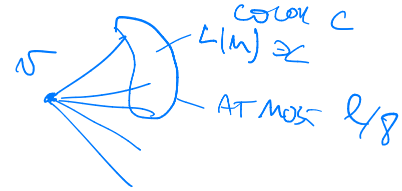
$\Rightarrow \exists$ COLORING WITH NO MONOCHROMATIC EDGE □

As a particular case of the above theorem, observe that the assumption is satisfied, if every edge is incident with at most 2^{k-3} others.

Theorem 5. *Let G be a graph and L list assignment with $|L(v)| \geq \ell > 0$ colors to every vertex v. Suppose that each color c appears in at most $\ell/8$ lists of the neighbours of each vertex. Then, G is L-colorable.*

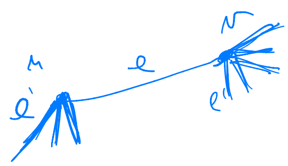
Proof. Color each vertex v of G independently and uniformrly with a color from $L(v)$. Thus each of its colors is chosen with probability $1/\ell$. For every edge $e = uv$ of G and every color $c \in L(u) \cap L(v)$, let $A_{c,e}$ be the event that both u and v are colored by c.

8: Finish the proof.



$P(A_{c,e}) = \frac{1}{\ell} \cdot \frac{1}{\ell} = \frac{1}{\ell^2}$

$A_{c,e}$ .. HOW MANY "DEPENDENT"?



$A_{c',e}$

u, v

$A_{c',e'}$

↳ ℓ candidates $\frac{\ell}{8}$ CANDIDATES $\frac{\ell \cdot \ell}{8} \cdot 2 = \frac{\ell^2}{4} = d$

$c \in e'$

$4 \cdot p \cdot d = 4 \cdot \frac{1}{\ell^2} \cdot \frac{\ell^2}{4} = 1$

$\Rightarrow \exists$ COLORING WITH NO BAD EVENT $\Rightarrow$ IT IS PROPER. □

Let $A_1, \dots, A_n$ be events in some probability space. A directed graph G with vertices $\{1, \dots, n\}$ is called a *dependency* graph, if each event A_i is mutually independent of all events A_j for which there is no oriented edge (i, j) in G. Notice that the dependency graph is not necessarily uniquely determined.

Theorem 6 (Asymmetric Lovász Local Lemma)**.** *Let be $A_1, \dots, A_n$ be events and $D = (V, E)$ dependency graph of these events. For each $i \in \{1, \dots, n\}$, let $x_i \in [0, 1)$ be real numbers for which*

$$\Pr(A_i) \le x_i \prod_{ij \in E} (1 - x_j).$$

Then,

$$\Pr(\bar{A}_1 \cap \bar{A}_2 \cap \cdots \cap \bar{A}_n) \ge \prod_{i=1}^{n} (1 - x_i) > 0.$$

9: Prove the symmetric version using the asymmetric version. Hint: Pick $x_i = \frac{1}{d+1} < 1$